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The Impact of $|\Delta I| = 5/2$ Transitions in K $\to \pi\pi$ Decays

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Abstract

We consider the impact of isospin violation on the analysis of $K \to \pi\pi$ decays. We scrutinize, in particular, the phenomenological role played by the additional weak amplitude, of $|\Delta I| = 5/2$ in character, incurred by the presence of isospin violation. We show that Watson's theorem is appropriate in $\mathcal{O}(m_d - m_u)$, so that the inferred $\pi - \pi$ phase shift at $\sqrt{s} = m_K$ determines the strong phase difference between the $I = 0$ and $I = 2$ amplitudes in $K \to \pi\pi$ decay. We find the magnitude of the $|\Delta I| = 5/2$ amplitude thus implied by the empirical branching ratios to be larger than expected from estimates of isospin-violating strong and electromagnetic effects. We effect a new determination of the octet and 27-plet coupling constants with strong-interaction isospin violation and with electromagnetic effects, as computed by Cirigliano, Donoghue, and Golowich, and find that we are unable to resolve the difficulty. Exploring the role of $|\Delta I| = 5/2$ transitions in the CP-violating observable ϵ'/ϵ , we determine that the presence of a $|\Delta I| = 5/2$ amplitude impacts the empirical determination of ω, the ratio of the real parts of the $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes, and that it generates a decrease in the estimation of ϵ'/ϵ .

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1 Introduction

In the limit of isospin symmetry, the decay of a kaon, with isospin $I_i = 1/2$, into two pions, with isospin $I_f = 0$ or $I_f = 2$, is mediated by either $|\Delta I| = 1/2$ or $|\Delta I| = 3/2$ weak transitions. The analysis of $K \to \pi\pi$ branching ratios in this limit indicates that the $|\Delta I| = 1/2$ amplitude exceeds the $|\Delta I| = 3/2$ amplitude by a factor of roughly twenty. A detailed understanding of this large enhancement, termed the "$|\Delta I| = 1/2$ rule," has proven elusive, although recently the subject has received much attention [1]. However, another, potentially related, puzzle remains. Unitarity and CPT invariance, in concert with isospin symmetry, predicts that the strong phase difference between the $I_f = 2$ and $I_f = 0$ amplitudes in $K \to \pi\pi$ decay should equal that of the $I = 2$ and $I = 0$ amplitudes in s-wave $\pi\pi$ scattering. The analysis of the $K \to \pi\pi$ branching ratios, using isospin-symmetric amplitudes but physical phase space, indicates, however, that this is not the case. Specifically, the strong phase difference inferred from $K \to \pi\pi$ decays is $\delta_0 - \delta_2 = 56.6^\circ \pm 4.5^\circ$ [2], whereas that from s-wave, $\pi\pi$ scattering at the kaon mass is $\delta_0 - \delta_2 = 45^\circ \pm 6^\circ$ [2, 3].

It is our purpose to examine how isospin-violating effects impact this apparent discrepancy. The u and d quarks differ both in their charges and masses, so that the symmetry of the $K \to \pi\pi$ decay amplitudes under u and d quark exchange is merely approximate. In specific, if we continue to use the labels "$I_f = 0$" and "$I_f = 2$" to denote the combinations of $K \to \pi\pi$ amplitudes which correspond to $\pi\pi$ final states of definite isospin in the isospin-perfect limit, then in this basis, the weak transitions are of $|\Delta I| = 1/2$, $3/2$, and $5/2$ in character. The violation of isospin symmetry thus generates an additional amplitude with $|\Delta I| = 5/2$. Such effects can modify the $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ amplitudes as well, though the large empirical enhancement of the $|\Delta I| = 1/2$ amplitude relative to the $|\Delta I| = 3/2$ amplitude found in the isospin-conserving analysis suggests that isospin-violating contributions built on the former are of greater phenomenological significance. Indeed, it has long been suspected that isospin-breaking effects contaminate the extracted ratio of $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes in a non-trivial way, precisely as isospin violation in the "large" $|\Delta I| = 1/2$ amplitude generates a contribution of $|\Delta I| = 3/2$ in character — and as the scale of strong interaction isospin violation, $(m_d - m_u)/m_s$, is crudely commensurate with that of the ratio determined in an isospin-perfect analysis. Indeed, including $m_d \neq m_u$ effects in a leading-order chiral analysis makes the "true" ratio of $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes some 30% smaller [4, 5]. We extract the $|\Delta I| = 1/2$, $3/2$, and $5/2$ amplitudes from the empirical $K \to \pi\pi$ branching ratios and then proceed to examine what solutions for the "true" $|\Delta I| = 1/2$ and $3/2$ amplitudes may emerge.

Interestingly, these considerations impact the Standard Model (SM) estimate of ϵ'/ϵ as well, for in standard practice the empirical value of the ratio of the real parts of the $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes is used, in concert with a "short-distance" determination of the amplitudes' imaginary parts, to determine ϵ'/ϵ in the SM [6, 7]. Isospin violation plays an important role in the analysis

of ϵ'/ϵ, for it modifies the cancellation of the imaginary to real part ratios in the $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ $K \to \pi\pi$ amplitudes in a significant manner [8, 9, 10, 11, 12]. The value of ω, the ratio of the real parts of the $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes, used, however, emerges from an analysis of $K \to \pi\pi$ branching ratios [13, 14], under the assumption that isospin symmetry is perfect. Thus we also explore the connection between isospin violation in $\mathrm{Re}\,(\epsilon'/\epsilon)$ and isospin violation in the $K \to \pi\pi$ branching ratios. We determine that the standard practice suffices to leading order in isospin violation if $|\Delta I| = 5/2$ transitions can be neglected. The $|\Delta I| = 5/2$ transitions enter differently in charged kaon and neutral kaon decays, and as the value of ω incorporated is derived, in part, from the $K^+ \to \pi^+\pi^0$ branching ratio, the value of ω must be adjusted for $|\Delta I| = 5/2$ effects in order to estimate ϵ'/ϵ . This decreases the value of ϵ'/ϵ and adds to its uncertainty as well.

We begin by considering the constraints that unitarity and time-reversal invariance place on the parametrization of the $K \to \pi\pi$ amplitudes in the presence of strong-interaction isospin violation. We consider exclusively $m_d \neq m_u$ effects as electromagnetic effects are considered in Ref. [15]. With an appropriate parametrization in place, we consider the phenomenological analysis of the $K \to \pi\pi$ branching ratios, extracting the amplitudes associated with the possible weak transitions and comparing these results with a chiral analysis. We then turn to ϵ'/ϵ and consider how isospin-violating effects in the branching ratios are related to those in ϵ'/ϵ .

2 Unitarity Constraints

We seek to determine what constraints may be brought to bear on the parametrization of the $K \to \pi\pi$ amplitudes in the presence of isospin violation. To this end enters Watson's theorem. We note that in the isospin-perfect limit, unitarity, and CPT invariance yields [16]

$$\begin{aligned} \langle(\pi\pi)_I|\mathcal{H}_W|K^0\rangle &= iA_I \exp(i\delta_I) \\ \langle(\pi\pi)_I|\mathcal{H}_W|\overline{K^0}\rangle &= -iA_I^* \exp(i\delta_I)\,, \end{aligned} \tag{1}$$

where $\mathcal{H}_W$ is the effective weak Hamiltonian for kaon decays. The amplitude A_I is such that $A_I = |A_I|\exp(i\xi_I)$, where ξ_I is the weak phase associated with the decay to the final state of isospin I, and δ_I is the phase associated with s-wave π-π scattering of isospin I.

In the limit of isospin symmetry, Bose statistics requires that two s-wave pions have either $I = 0$ or $I = 2$. To relate the isospin states to the physical states, we use the isospin decomposition [17]

$$\begin{aligned} |\pi^+\pi^-\rangle &\propto |(\pi\pi)_0\rangle + \frac{1}{\sqrt{2}}|(\pi\pi)_2\rangle \\ |\pi^0\pi^0\rangle &\propto |(\pi\pi)_0\rangle - \sqrt{2}|(\pi\pi)_2\rangle\ . \end{aligned} \tag{2}$$

Using Watson's theorem, Eq. (1), and including isospin-violating effects, we have the parametrization

$$
\begin{aligned}
A_{K^0\to\pi^+\pi^-} \equiv \langle \pi^+\pi^-|\mathcal{H}_W|K^0\rangle &= i(A_0 e^{i\delta_0} + \frac{1}{\sqrt{2}}A_2 e^{i\delta_2} + A_{\rm IB}^{+-} e^{i\delta_{+-}}) \\
A_{K^0\to\pi^0\pi^0} \equiv \langle \pi^0\pi^0|\mathcal{H}_W|K^0\rangle &= i(A_0 e^{i\delta_0} - \sqrt{2}A_2 e^{i\delta_2} + A_{\rm IB}^{00} e^{i\delta_{00}}) , \\
A_{K^+\to\pi^+\pi^0} \equiv \langle \pi^+\pi^0|\mathcal{H}_W|K^+\rangle &= i(\frac{3}{2}A_2 e^{i\delta_2} + A_{\rm IB}^{+0} e^{i\delta_{+0}}) ,
\end{aligned}
\tag{3}
$$

where the isospin-violating contributions are denoted by the subscript "IB" and include a weak phase, e.g., $A_{\rm IB}^{00} = |A_{\rm IB}^{00}|e^{i\xi_{00}}$. The strong phases δ_{00}, δ_{+-}, and δ_{+0} are, as yet, idiosyncratic to $K \to \pi\pi$ decay. As A_0 and A_2 are reflective of the amplitudes in the isospin-perfect limit, they are generated by $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ weak transitions, respectively.

We wish to examine what further constraints may be placed on Eq. (3). It follows from unitarity that a transition matrix T satisfies the relation

$$
T^\dagger T = i(T^\dagger - T) , \tag{4}
$$

where the S matrix can be written as $S = 1+iT$ and unitarity is the condition $S^\dagger S = 1$. We consider $K \to \pi\pi$ decays, so that the final-state phases of interest are generated through π-π scattering. In the presence of isospin violation, the isospin-perfect basis of Eq. (2) continues to prove convenient, as the possibility of $\pi^+\pi^- \leftrightarrow \pi^0\pi^0$ through strong rescattering makes the "physical" basis awkward. The label "I," however, need only correspond to the isospin of the final-state pions in the isospin-perfect limit. We begin by considering $K^0 \to (\pi\pi)_I$ decays and find, upon insertion of *all* possible intermediate states F:

$$
\sum_F \langle (\pi\pi)_I|T^\dagger|F\rangle\langle F|T|K^0\rangle = i(\langle (\pi\pi)_I|T^\dagger|K^0\rangle - \langle (\pi\pi)_I|T|K^0\rangle) . \tag{5}
$$

Note that F denotes the set of states physically accessible in K decay and thus includes the $(\pi\pi)_I$ states defined in Eq. (2), as well as $\pi^+\pi^-\gamma$, $\gamma\gamma$, and 3π states. In the isospin-perfect limit, only the $F = (\pi\pi)_I$ term in the sum contributes. The inclusion of electromagnetic effects, however, complicates matters, as additional states may contribute to the sum in Eq. (5). The most significant of the modes with photons or leptons in the final state is $K_S^0 \to \pi^+\pi^-\gamma$; let us continue to neglect such electromagnetic isospin-violating effects and investigate the effects of strong-interaction isospin violation. We also neglect the 3π intermediate state appearing in Eq. (5) because the $\langle(\pi\pi)_I|T|3\pi\rangle$ transition amplitude with $J = 0$ violates not only G-parity but P as well. Note that the spatial component of the $J = 0$ 3π state is even under P, so that the $J = 0$ 3π state is of odd parity [18]. We work to leading order in the weak interaction, so that $\langle 2\pi|T|3\pi\rangle$ is mediated by strong rescattering and thus vanishes for $J = 0$ states, as the strong interaction conserves parity. At the energies appropriate to kaon decay, the strong scattering in the $(\pi\pi)_I$ final state is described by a pure

phase, as the empirical inelasticity parameters are unity [19], so that in the isospin-perfect limit we can write

$$\mathbf{S} = \begin{pmatrix} e^{2i\delta_0} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} . \tag{6}$$

Thus if isospin is a perfect symmetry, only $F = (\pi\pi)_I$ contributes to the sum and one recovers the usual parametrization

$$\begin{aligned} \langle (\pi\pi)_I | T | K^0 \rangle &= iA_I \exp(i\delta_I) \\ \langle (\pi\pi)_I | T | \overline{K^0} \rangle &= -iA_I^* \exp(i\delta_I) , \end{aligned} \tag{7}$$

noting by CPT symmetry that $\langle (\pi\pi)_I | T^\dagger | K^0 \rangle = (\langle (\pi\pi)_I | T | \overline{K^0} \rangle)^*$.

We now turn to the consideration of isospin-violating effects. The S-matrix appropriate to the $\pi\pi$ final states with zero net charge is characterized, in general, by eight real parameters. Unitarity, however, yields three distinct constraints, and time-reversal invariance yields two more, so that the S-matrix can contain at most three real parameters. We have seen from the explicit form of S-matrix in the isospin-perfect limit that it is characterized by precisely two parameters, δ_0 and δ_2 — and thus the third parameter permitted by unitarity and time-reversal invariance must be at least of $\mathcal{O}(m_d - m_u)$, or of $\mathcal{O}(\alpha)$. As electromagnetic effects in the $K \to \pi\pi$ phases are studied in Ref. [15], we focus on $m_d \neq m_u$ effects.

We parametrize the S-matrix in the presence of isospin violation as [20]

$$\mathbf{S} = \begin{pmatrix} e^{i\bar{\delta}_0} & 0 \\ 0 & e^{i\bar{\delta}_2} \end{pmatrix} \begin{pmatrix} \cos 2\kappa & i\sin 2\kappa \\ i \sin 2\kappa & \cos 2\kappa \end{pmatrix} \begin{pmatrix} e^{i\bar{\delta}_0} & 0 \\ 0 & e^{i\bar{\delta}_2} \end{pmatrix} \tag{8}$$

where the third S-matrix parameter is denoted by κ. Note that if $\kappa = 0$ then $\bar{\delta}_I = \delta_I$, where δ_I denote the strong phases of the isospin-perfect limit. In the presence of isospin violation we continue to use Eq. (2) to define the $|(\pi\pi)_I\rangle$ states used in Eq. (8). The parameter κ is sensitive to $m_d \neq m_u$ effects in the strong chiral Lagrangian, as well as to electromagnetic effects. Explicit calculation shows that all strong-interaction isospin-violating effects in $\pi\pi$ scattering are at least of $\mathcal{O}((m_d - m_u)^2)$ in $\mathcal{O}(p^4)$ in the chiral expansion [21]. This result persists to all orders in chiral perturbation theory; let us turn to an explicit demonstration of this point.

Isospin violation in the S-matrix element for 2-to-2 $\pi\pi$ scattering can occur in either the truncated, connected Green function itself or in the external π legs. The latter source of isospin violation emerges as in $\mathcal{O}(m_d - m_u)$ the π^0 and η fields mix. Diagonalizing the neutral, non-strange meson states of the strong chiral Lagrangian yields, in $\mathcal{O}(p^2)$, e.g., yields the "physical" π^0 state in terms of the pseudoscalar octet fields π^0 and η [22]:

$$\left(\pi^0\right)_{\rm phys} = \pi^0 + \frac{\sqrt{3}}{4} \left(\frac{m_d - m_u}{m_s - \hat{m}} \right) \eta + \mathcal{O}((m_d - m_u)^2) , \tag{9}$$

where $\hat{m} = (m_d + m_u)/2$. An analogous formula exists in $\mathcal{O}(p^4)$ [22]. Thus isospin violation in an external π leg is realized as an η admixture in the physical π^0 state. In the pseudoscalar octet, or "isospin-perfect," basis we have adopted thus far, an $\mathcal{O}(m_d - m_u)$ interaction converts the isovector π^0 into a isoscalar η. Thus in $\mathcal{O}(m_d - m_u)$ the truncated, connected Green function arising from isospin violation in an external π leg contains one η and three π fields. Note that the decay $\eta \to \pi\pi\pi$ is forbidden by Bose symmetry in the isospin-symmetric limit, $m_d = m_u$, so that the truncated, connected Green function of interest must be at least of $\mathcal{O}(m_d - m_u)$. Including the $(m_d - m_u)$ "penalty" required to convert the η to a physical π^0, one finds that isospin-violating effects arising from the external legs start in $\mathcal{O}((m_d - m_u)^2)$. One can also show that the $m_d \neq m_u$ effects in the truncated, connected Green function associated with the 2-to-2 scattering of isovector pions also start in $\mathcal{O}((m_d - m_u)^2)$. Following the "spurion" formulation [23], a transition matrix element with SU(2) violation must have the same properties as a SU(2)-conserving transition matrix element containing a spurion, a fictitous particle which carries, in this case, the quantum numbers of the π^0 and a factor of $(m_d - m_u)$. Thus the spurion and the π are both of negative G-parity, so that a transition of form

$$\text{(even number of pions)} \Longleftrightarrow \text{(even number of pions + 1 spurion)} \tag{10}$$

is forbidden by G-parity and does not occur [24]. Note, however, that a transition of form

$$\text{(even number of pions)} \Longleftrightarrow \text{(even number of pions + 2 spurions)} \tag{11}$$

is permitted by G-parity, so that all isospin-violating effects in π-π scattering are of $\mathcal{O}((m_d - m_u)^2)$. Analyzing Eq. (8), this result implies that

$$\bar{\delta}_I - \delta_I \sim \mathcal{O}((m_d - m_u)^2) \quad ; \quad \kappa \sim \mathcal{O}((m_d - m_u)^2)\;. \tag{12}$$

so that $\kappa = 0$ in $\mathcal{O}(m_d - m_u)$.

Using Eq. (8) to incorporate isospin violation in $K \to \pi\pi$ decays, we find that Eq. (5) thus becomes

$$\begin{pmatrix} 1 - e^{-2i\bar{\delta}_0}\cos 2\kappa & -ie^{-i(\bar{\delta}_0+\bar{\delta}_2)}\sin 2\kappa \\ -ie^{-i(\bar{\delta}_0+\bar{\delta}_2)}\sin 2\kappa & 1 - e^{-2i\bar{\delta}_2}\cos 2\kappa \end{pmatrix} \begin{pmatrix} \langle(\pi\pi)_0|T|K^0\rangle \\ \langle(\pi\pi)_2|T|K^0\rangle \end{pmatrix} = \begin{pmatrix} \langle(\pi\pi)_0|T|K^0\rangle - \langle(\pi\pi)_0|T^\dagger|K^0\rangle \\ \langle(\pi\pi)_2|T|K^0\rangle - \langle(\pi\pi)_2|T^\dagger|K^0\rangle \end{pmatrix} \tag{13}$$

Following the parametrization of Eq. (7), we have in the presence of isospin violation

$$\begin{aligned} \langle(\pi\pi)_I|T|K^0\rangle &= iA_I \exp(i\tilde{\delta}_I) \\ \langle(\pi\pi)_I|T|\overline{K^0}\rangle &= -iA_I^* \exp(i\tilde{\delta}_I)\,, \end{aligned} \tag{14}$$

where $\tilde{\delta}_I$, the strong phase of the $K \to \pi\pi$ decay amplitude, is related to the strong phase of $\pi\pi$ scattering, given in Eq. (8), as per Eq. (13). We thus have

$$\begin{pmatrix} 1 - e^{-2i\bar{\delta}_0}\cos 2\kappa & -ie^{-i(\bar{\delta}_0+\bar{\delta}_2)}\sin 2\kappa \\ -ie^{-i(\bar{\delta}_0+\bar{\delta}_2)}\sin 2\kappa & 1 - e^{-2i\bar{\delta}_2}\cos 2\kappa \end{pmatrix} \begin{pmatrix} A_0 e^{i\tilde{\delta}_0} \\ A_2 e^{i\tilde{\delta}_2} \end{pmatrix} = 2i \begin{pmatrix} A_0 \sin\tilde{\delta}_0 \\ A_2 \sin\tilde{\delta}_2 \end{pmatrix} \tag{15}$$

Note that if the channel-coupling parameter κ were zero, then $\tilde{\delta}_I = \bar{\delta}_I = \delta_I$, and the strong-phase in the $K \to \pi\pi$ decay amplitude would be that of $\pi\pi$ scattering, analyzed in the isospin-perfect limit. Defining

$$\Delta_I \equiv \bar{\delta}_I - \tilde{\delta}_I \;, \tag{16}$$

so that $\Delta_I = 0$ were $\kappa = 0$, and rearranging the upper component of Eq. (15), we find

$$e^{-2i\Delta_0}\cos(2\kappa) - 1 = -i\frac{A_2}{A_0}e^{-i(\Delta_0+\Delta_2)}\sin(2\kappa) \;. \tag{17}$$

Using the lower component of Eq. (15) yields Eq. (17) with the isospin subscripts switched, $0 \leftrightarrow 2$. As $\kappa \to 0$, $\Delta_I \to 0$ as well, and we find

$$\Delta_0 = \frac{A_2}{A_0}\kappa + \mathcal{O}(\kappa^2) \quad ; \quad \Delta_2 = \frac{A_0}{A_2}\kappa + \mathcal{O}(\kappa^2) \;, \tag{18}$$

implying $\Delta_2 \gg \Delta_0$ and $\Delta_0\Delta_2 \sim \kappa^2$. Eliminating A_2/A_0 from Eq. (17) and its $0 \leftrightarrow 2$ counterpart yields a relation purely in terms of Δ_I and κ:

$$\cos(2\kappa)\cos(\Delta_0 - \Delta_2) = \cos(\Delta_0 + \Delta_2) \;. \tag{19}$$

Alternatively, one can eliminate κ to find

$$A_2^2\sin(2\Delta_2) = A_0^2\sin(2\Delta_0) \;. \tag{20}$$

With Eqs. (12) and (18) we have that $\tilde{\delta}_I - \delta_I$ is no larger than

$$\tilde{\delta}_I - \delta_I \sim \mathcal{O}((m_d - m_u)^2) \;. \tag{21}$$

Thus in $\mathcal{O}(m_d - m_u)$ the channel-coupling parameter $\kappa = 0$ and $\tilde{\delta}_I = \delta_I$, so that the parametrization of Eq. (7) is appropriate in the presence of strong-interaction isospin violation as well. However, if electromagnetic effects were included, one would expect $\kappa \sim \mathcal{O}(\alpha)$, and with $A_2/A_0 \sim 1/20$, one finds $|\Delta_2| \sim |\tilde{\delta}_2 - \delta_2| \sim \mathcal{O}(10^\circ)$ [25], commensurate with the explicit estimate of 4.5° in $\mathcal{O}(e^2p^0)$ in Ref. [15].

We consider how our results generalize to the case of $K^+ \to \pi^+\pi^0$ decays as well, for these decays are needed to isolate the $|\Delta I| = 5/2$ amplitude. In the case of charged $K \to \pi\pi$ decays, Eq. (5) becomes

$$\sum_F \langle(\pi\pi)_{I^+}|T^\dagger|F\rangle\langle F|T|K^+\rangle = i(\langle(\pi\pi)_{I^+}|T^\dagger|K^+\rangle - \langle(\pi\pi)_{I^+}|T|K^+\rangle) \;, \tag{22}$$

where we now explicitly denote the isospin I, $I_3 = 1$ final state by "$(\pi\pi)_{I^+}$". Charge is conserved so that Eq. (22) is diagonal in I_3. Neglecting the 3π and electromagnetic intermediate states, we thus have

$$\langle(\pi\pi)_{2^+}|T^\dagger|(\pi\pi)_{2^+}\rangle\langle(\pi\pi)_{2^+}|T|K^+\rangle = i(\langle(\pi\pi)_{2^+}|T^\dagger|K^+\rangle - \langle(\pi\pi)_{2^+}|T|K^+\rangle) \;. \tag{23}$$

By crossing symmetry, our prior analysis of isospin violation in $\pi\pi$ scattering is germane to this case as well, so that we conclude that strong-interaction isospin-violating effects in $\langle(\pi\pi)_{2^+}|T^\dagger|(\pi\pi)_{2^+}\rangle$ are of $\mathcal{O}((m_d - m_u)^2)$. Thus we write $\langle(\pi\pi)_{2^+}|T^\dagger|(\pi\pi)_{2^+}\rangle = -i(1 - e^{-2i\delta_2})$, or finally

$$\langle(\pi\pi)_{2^+}|T|K^+\rangle = iA_{2^+}e^{i\delta_2} \ , \tag{24}$$

so that, with the neglect of electromagnetic effects, the strong phase in this channel is related to that of the $I = 2$ amplitude comprised of charge-neutral final states. It is worth noting that the phase of Eq. (24) is evaluated at $\sqrt{s} = m_{K^+}$, whereas the phases of $K^0 \to \pi\pi$ decay is evaluated at $\sqrt{s} = m_{K^0}$. However, this small difference is without practical consequence, for the phase of Eq. (24) does not appear in the $K^+ \to \pi\pi$ branching ratio.

We have thus demonstrated in $\mathcal{O}(m_d - m_u)$ that the strong phases of the $K \to \pi\pi$ amplitudes are those of $\pi\pi$ scattering in the isospin-perfect limit. Generally, $m_d \neq m_u$ effects permit amplitudes of $|\Delta I| = 1/2$, $3/2$, and $5/2$ in character, so that the parametrization of Eq. (3) can be rewritten as

$$\begin{aligned} A_{K^0\to\pi^+\pi^-} &= i((A_0 + \delta A_{1/2})e^{i\delta_0} + \frac{1}{\sqrt{2}}(A_2 + \delta A_{3/2} + \delta A_{5/2})e^{i\delta_2}) \\ A_{K^0\to\pi^0\pi^0} &= i((A_0 + \delta A_{1/2})e^{i\delta_0} - \sqrt{2}(A_2 + \delta A_{3/2} + \delta A_{5/2})e^{i\delta_2}) \\ A_{K^+\to\pi^+\pi^0} &= i(\frac{3}{2}(A_2 + \delta A_{3/2}) - \delta A_{5/2})e^{i\delta_2} \ , \end{aligned} \tag{25}$$

in $\mathcal{O}(m_d - m_u)$, where $\delta A_{|\Delta I|}$ denotes the amplitude contributions induced exclusively by isospin violation. Note that the parametrization of the charge-conjugate decays follows from Eq. (14). The $\delta A_{1/2}$ and $\delta A_{3/2}$ contributions are each generated by both $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ weak transitions. The presence of a $\delta A_{5/2}$ contribution — the "new" amplitude — is signalled by the inequality $(A_{K^0\to\pi^+\pi^-} - A_{K^0\to\pi^0\pi^0})/\sqrt{2} - A_{K^+\to\pi^+\pi^0} \neq 0$ [26].

3 Phenomenology of $K \to \pi\pi$ Decays

We now wish to determine the relative magnitude of the various amplitudes in Eq. (25) predicated by the measured $K \to \pi\pi$ branching ratios and by the inferred π-π phase shifts. To this end, we consider the following ratios of reduced transition rates:

$$R_1 = \frac{\gamma(K_S^0 \to \pi^+\pi^-)}{\gamma(K_S^0 \to \pi^0\pi^0)} \tag{26}$$

and

$$R_2 = \frac{2\gamma(K^+ \to \pi^+\pi^0)}{\gamma(K_S^0 \to \pi^+\pi^-) + \gamma(K_S^0 \to \pi^0\pi^0)} \ , \tag{27}$$

where $\gamma(K \to \pi_1\pi_2)$, the reduced transition rate, is related to the partial width $\Gamma(K \to \pi_1\pi_2)$ via

$$\Gamma(K \to \pi_1\pi_2) = \frac{\sqrt{(m_K^2 - (m_{\pi_1} + m_{\pi_2})^2)(m_K^2 - (m_{\pi_1} - m_{\pi_2})^2)}}{16\pi m_K^3} \gamma(K \to \pi_1\pi_2) \; . \tag{28}$$

We use the physical π and K masses in extracting $\gamma(K \to \pi\pi)$, and neglect any final-state Coulomb corrections as they are electromagnetic effects. The reduced transition rates are simply related to the absolute squares of the amplitudes we have considered previously, so that

$$\begin{aligned} R_1 &= \frac{2|A_{K_S^0 \to \pi^+\pi^-}|^2}{|A_{K_S^0 \to \pi^0\pi^0}|^2} \\ R_2 &= \frac{2|A_{K^+ \to \pi^+\pi^0}|^2}{2|A_{K_S^0 \to \pi^+\pi^-}|^2 + |A_{K_S^0 \to \pi^0\pi^0}|^2} \; . \end{aligned} \tag{29}$$

Using the parametrization of Eq. (25), noting $K_S = (K^0 - \overline{K^0})/\sqrt{2}$ with $CP(K^0) = -\overline{K^0}$, while ignoring CP violation and weak phases, yields

$$2\sqrt{\frac{R_2}{3}} = \pm(x - \frac{2}{3}y) \; ; \tag{30}$$

$$\begin{aligned} \frac{R_1}{2} &= \frac{1 + \sqrt{2}(x+y)\cos(\delta_2 - \delta_0) + (x+y)^2/2}{1 - 2\sqrt{2}(x+y)\cos(\delta_2 - \delta_0) + 2(x+y)^2} \\ &= 1 + 3\sqrt{2}(x+y)\cos(\delta_2 - \delta_0) + (12\cos^2(\delta_2 - \delta_0) - 3/2)x^2 + \mathcal{O}(xy, x^3, y^2) \; , \end{aligned} \tag{31}$$

where, working consistently to leading order in isospin violation, we have

$$\begin{aligned} x &\equiv \frac{A_2 + \delta A_{3/2}}{A_0 + \delta A_{1/2}} \approx \frac{A_2}{A_0} + \frac{\delta A_{3/2}}{A_0} - \frac{A_2}{A_0}\frac{\delta A_{1/2}}{A_0} \; , \\ y &\equiv \frac{\delta A_{5/2}}{A_0 + \delta A_{1/2}} \approx \frac{\delta A_{5/2}}{A_0} \; . \end{aligned} \tag{32}$$

The ratio x is A_2/A_0 in the isospin-perfect limit, whereas the ratio y is non-zero only in the presence of isospin violation. We anticipate that a $\delta A_{5/2}$ contribution is generated either by strong-interaction isospin violation in concert with a $|\Delta I| = 3/2$ weak transition, or by electromagnetic effects in concert with a $|\Delta I| = 1/2$ weak transition. We thus expect the hierarchy $x \gg x^2, y \gg x^3, xy, y^2$, which is reflected in the terms retained in Eq. (31). Note that it is appropriate to continue to work to leading order in isospin violation after the inclusion of the $|\Delta I| = 5/2$ contributions, as crudely $|A_2/A_0| \sim 5\%$ — this follows from Eq. (30) if $y = 0$ — whereas isospin violation is a $\sim 1\%$ effect.

Let now proceed to determine x and y. We determine R_1 and R_2 using the "our fit" branching ratios and ancillary empirical data in Ref. [27] and plot the x and y resulting from Eqs. (30,31) as a function of $\delta_0 - \delta_2$ in Fig. 1. Note that $\cos(\delta_2 - \delta_0) > 0$ and $R_1/2 > 1$, so that Eq. (31) implies that $x + y > 0$. As we assume $x \gg y$, then $x > 0$ as well, and we choose the + sign in Eq. (30) in

what follows [28]. Moreover, we pick the root of the quadratic equation consistent with $A_0 > A_2$. We affect these choices in order to recover the qualitative features of the analysis performed in the $m_d \to m_u$ limit. The errors in x and y arise from the empirical errors, assuming all the errors are uncorrelated. The vertical dashed lines enclose the phase shift difference $\delta_0 - \delta_2 = 45^\circ \pm 6^\circ$ [2], whereas the vertical dot-dashed lines enclose $\delta_0 - \delta_2 = 45.2^\circ \pm 1.3^\circ \pm\,^{4.5^\circ}_{1.6^\circ}$ [3] at 68% C.L. We omit explicit use of this latter value in what follows as it is comparable to the result of Ref. [2]. Table 1 shows the specific values of x and y, with their associated errors, which emerge from combining the empirical values of R_1 and R_2 with the values of $\delta_0 - \delta_2$ from various sources. Note that we use the $\delta_0 - \delta_2$ phase shift as extracted in the isospin-symmetric limit, as strong-interaction isospin-violating effects enter merely in $\mathcal{O}((m_d - m_u)^2)$ and as the electromagnetically generated $K \to \pi\pi$

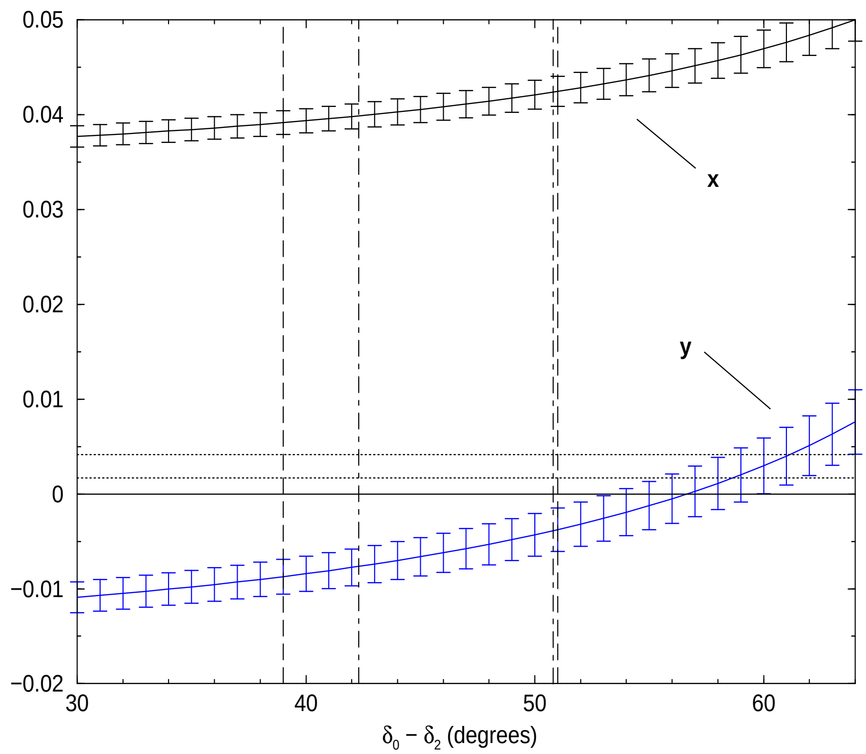

Figure 1: The values of x and y resulting from Eqs. (30,31) as a function of $\delta_0 - \delta_2$. In the isospin-perfect limit $x = A_2/A_0$ and $y = 0$. The vertical dashed and dot-dashed lines enclose the results $\delta_0 - \delta_2 = 45^\circ \pm 6^\circ$ [2] and $\delta_0 - \delta_2 = 45.2^\circ \pm 1.3^\circ \pm\,^{4.5^\circ}_{1.6^\circ}$ [3], respectively, at 68% C.L. The two sets of vertical lines overlap at 51° — the dot-dashed line has been slightly off-set for presentation. The horizontal dashed line encloses the electromagnetic contribution to y as per the "dispersive matching" calculation of Table I of Ref. [15] at 68% C.L.

phase shifts appear to be small [15]. For estimates of electromagnetic effects in $\pi-\pi$ scattering, see Ref. [29].

Proceeding with the numerical analysis, we find a substantial value for $\delta A_{5/2}$, suggesting the phenomenological hierarchy $x \gg y \gg x^2, xy$. Specifically, we find

$$\delta A_{5/2}/(A_2+\delta A_{3/2}) \sim 20\% \;, \tag{33}$$

rather than the $\mathcal{O}(1\%)$ we might have anticipated from strong-interaction isospin violation. The extracted $\delta A_{5/2}$ amplitude is sensitive to the value of $\delta_0-\delta_2$ used; indeed, were $\delta_0-\delta_2 \sim 56.6^\circ$, then $\delta A_{5/2} \sim 0$. Moreover, if the errors in $\delta_0-\delta_2$ were consistently — and substantially — underestimated, our determined $\delta A_{5/2}$ could be made consistent with zero. In particular, if we were to increase the error in $\delta_0-\delta_2$ to realize this, we would find that we would require, e.g., $45^\circ \pm 16^\circ$. Such increases would reflect a severe inflation of the stated error bars and would seem unwarranted. It ought be realized that $\pi\pi$ phase shift information is largely inferred from associated production in πN reactions and that any possible theoretical systematic errors incurred through the choice of reaction model are not incorporated in the reported error estimates [30]. However, information on the $I=0$ $\pi\pi$ phase shift near threshold is also known from $K\to\pi\pi e\nu$ decay; this is consistent with the phase shift determined in πN reactions, albeit the errors are large [31]. Interestingly, the $e^+e^-\to\pi\pi$ and $\tau\to\pi\pi\nu$ data in the context of a Roy equation analysis of $\pi\pi$ scattering constrain the possible s-wave phase shifts rather significantly, yielding at $s=m_{K^0}^2$ that $\delta_0-\delta_2 = 45.2^\circ \pm 1.3^\circ \pm\, ^{4.5^\circ}_{1.6^\circ}$ [3]. This is commensurate with earlier determinations of $\delta_0-\delta_2$ [32, 33, 34, 13], noting Table 1, and encourages us to consider the consequences of our fit.

Table 1: The values of x and y resulting from Eqs. (30,31) using the phase shift differences, $\delta_0-\delta_2$, compiled from various sources. Note that in the isospin-perfect limit that $x=A_2/A_0$ and $y=0$. The values found for $|y|$ are roughly equal to α, suggesting an electromagnetic origin for y.

Ref.	$\delta_0-\delta_2$ (deg.)	x	y
[13]	41.4 ± 8.1	0.0396 ± 0.0022	-0.0080 ± 0.0033
[34]	42 ± 4	0.0398 ± 0.0016	-0.0077 ± 0.0024
[34] ("local fit")	42 ± 6	0.0398 ± 0.0019	-0.0077 ± 0.0028
[32]	44 ± 5	0.0403 ± 0.0019	-0.0070 ± 0.0028
[2]	45 ± 6	0.0405 ± 0.0021	-0.0066 ± 0.0032

Let us first compare our results with the $\delta A_{5/2}$ amplitude estimated to be induced by electromagnetism [15]. Using Eq. (48) and the "dispersive matching" estimate of Table I in Ref. [15], we find $y_{\rm em} \sim 0.0029$, suggesting that the computed electromagnetic effects are rather smaller and are of the wrong sign [35]. Indeed, this discrepancy prompts our consideration of strong-interaction

isospin-violating effects. In particular, were $y = 0$, then Eq. (31) would become

$$\frac{R_1}{2} = \frac{1 + \sqrt{2}x\cos(\delta_2 - \delta_0) + x^2/2}{1 - 2\sqrt{2}x\cos(\delta_2 - \delta_0) + 2x^2} = 1 + 3\sqrt{2}x\cos(\delta_2 - \delta_0) + (12\cos^2(\delta_2 - \delta_0) - 3/2)x^2 + \mathcal{O}(x^3) \ . \tag{34}$$

Using $\delta_0 - \delta_2 = 45^\circ$ [2] and Ref. [27] yields $x = 0.035$, whereas Eq. (30) in this limit would be

$$2\sqrt{\frac{R_2}{3}} = x \tag{35}$$

and yields $x = 0.045$ — this discrepancy is reconciled through the value of y we report in Table 1. The significance of y could be exacerbated by the parameters reported in Ref. [27], though excursions of several standard deviations are required to impact its value significantly [36].

We summarize this section with the following observations.

- The value of x is stable with respect to the various values of $\delta_0 - \delta_2$ reported in Table 1 — it varies merely at the 1% level.
- The value of y is rather more sensitive to $\delta_0 - \delta_2$. It apparently is of $\mathcal{O}(\alpha)$, rather than of $\mathcal{O}(\omega(m_d - m_u)/m_s)$ — and thus is rather larger than expected from the standpoint of strong-interaction isospin violation.

4 Isospin Violation and the $|\Delta I| = 1/2$ Rule

Our determined x and y may be connected to the amplitudes of the isospin-perfect limit, A_0 and A_2, via a computation of the $K \to \pi\pi$ amplitudes in chiral perturbation theory. The weak chiral Lagrangian in $\mathcal{O}(p^2)$ has two non-trivial terms, which transform as $(8_L, 1_R)$ and as $(27_L, 1_R)$ under $SU(3)_L \times SU(3)_R$, respectively [37]. We wish to determine their relative magnitude in the context of a calculation which is sensitive to $m_u \neq m_d$ effects, in order to assess the relative strength of the $(27_L, 1_R)$ and $(8_L, 1_R)$ transitions, that is, the ratio A_2/A_0. We believe that $m_u \neq m_d$ effects likely contribute to x in a significant manner [4, 5]. Ultimately we will also include the computed electromagnetic corrections of Ref. [15] as well, in order to determine A_2/A_0, for the numerical value of y is crudely an $\mathcal{O}(\alpha)$ effect.

In $\mathcal{O}(p^2)$, the $(8_L, 1_R)$ term in the weak, chiral Lagrangian generates exclusively $|\Delta I| = 1/2$ transitions, whereas the $(27_L, 1_R)$ term generates both $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ transitions. We have [38]

$$\mathcal{L}_W^{(2)} = -\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\Big[g_8\,(L_\mu L^\mu)_{23} + g_{27}^{(1/2)}\,(L_{\mu 13}L^\mu_{21} + L_{\mu 23}(4L^\mu_{11} + 5L^\mu_{22})) + g_{27}^{(3/2)}\,(L_{\mu 13}L^\mu_{21} + L_{\mu 23}(L^\mu_{11} - L^\mu_{22}))\Big] + \text{h.c.} \ , \tag{36}$$

where $L_\mu = -if_\pi^2 U D_\mu U^\dagger$ with $U = \exp(-i\vec{\lambda}\cdot\vec{\phi}(x))/f_\pi$[38]. The function $\vec{\phi}$ represents the octet of pseudo-Goldstone bosons. The low-energy constants $g_{27}^{(1/2)}$ and $g_{27}^{(3/2)}$ are associated with $|\Delta I| = 1/2$ and $3/2$ $(27_L, 1_R)$ transitions, respectively. We retain $g_{27}^{(1/2)}$ and $g_{27}^{(3/2)}$ as distinct entities as we anticipate the $\mathrm{SU}(3)_f$ relation $g_{27}^{(1/2)} = g_{27}^{(3/2)}/5$ is broken at higher orders in the weak chiral expansion — we will see what other features are required to incorporate the effects of higher-order terms in a systematic manner. No "weak mass" term occurs in leading order in the weak chiral Lagrangian [37], so that $m_u \neq m_d$ effects appear exclusively through π^0-η mixing, as realized in Eq. (9), and meson mass differences. In $\mathcal{O}(p^2)$ and to leading order in $(m_d - m_u)$, we have

$$
\begin{aligned}
A_{K^0\to\pi^+\pi^-} &= \sqrt{2}Ci\left(g_8 + g_{27}^{(1/2)} + g_{27}^{(3/2)} + \frac{2\epsilon_8}{\sqrt{3}}(g_8 + g_{27}^{(1/2)} + g_{27}^{(3/2)})\right) \\
A_{K^0\to\pi^0\pi^0} &= \sqrt{2}Ci\left(g_8 + g_{27}^{(1/2)} - 2g_{27}^{(3/2)} - \frac{2\epsilon_8}{\sqrt{3}}(5g_{27}^{(1/2)} - g_{27}^{(3/2)})\right) \\
A_{K^+\to\pi^+\pi^0} &= Ci\left(3g_{27}^{(3/2)} + \frac{\epsilon_8}{\sqrt{3}}(2g_8 + 12g_{27}^{(1/2)} - 3g_{27}^{(3/2)})\right) \ ,
\end{aligned}
\tag{37}
$$

where $\epsilon_8 = \sqrt{3}/4((m_d - m_u)/(m_s - \hat{m}))$ and $C = -(G_F/\sqrt{2})V_{ud}V_{us}^* f_\pi (m_s - \hat{m})B_0$, and $(m_s - \hat{m})B_0 = m_K^2 - m_\pi^2$ in the isospin-perfect limit. We thus recover

$$
\begin{aligned}
A_0 + \delta A_{1/2} &= C\left(\sqrt{2}(g_8 + g_{27}^{(1/2)}) + \frac{2}{3}\sqrt{\frac{2}{3}}\epsilon_8(2g_8 - 3g_{27}^{(1/2)} + 3g_{27}^{(3/2)})\right) \\
A_2 + \delta A_{3/2} &= C\left(2g_{27}^{(3/2)} + \frac{2}{\sqrt{3}}\epsilon_8(\frac{2}{3}g_8 + 4g_{27}^{(1/2)} - \frac{3}{5}g_{27}^{(3/2)})\right) \\
\delta A_{5/2} &= \frac{2\sqrt{3}}{5}C\epsilon_8 g_{27}^{(3/2)}
\end{aligned}
\tag{38}
$$

and

$$
x = \frac{\sqrt{2}r^{(3/2)}}{1 + r^{(1/2)}}\left(1 - \frac{2}{3\sqrt{3}}\epsilon_8\frac{(2 + 3(r^{(3/2)} - r^{(1/2)}))}{1 + r^{(1/2)}}\right) + \frac{\epsilon_8}{15}\sqrt{\frac{2}{3}}\frac{(10 - 9r^{(3/2)} + 60r^{(1/2)})}{1 + r^{(1/2)}}
\tag{39}
$$

$$
y = \frac{\sqrt{6}}{5}\frac{\epsilon_8 r^{(3/2)}}{1 + r^{(1/2)}} \ ,
\tag{40}
$$

where $r^{(1/2)} \equiv g_{27}^{(1/2)}/g_8$ and $r^{(3/2)} \equiv g_{27}^{(3/2)}/g_8$. We will allow $r^{(1/2)} \neq r^{(3/2)}/5$ in our fits as well, in order to ape the inclusion of higher-order effects in the weak chiral Lagrangian. Were the fits in the isospin-symmetric limit a reasonable estimate of the low-energy constants, so that Eq.(30) yields $|A_2/A_0| \sim 0.045$[13], we would expect $|y|$ to be roughly $1.7\cdot 10^{-4}$, as $(m_s - \hat{m})/(m_d - m_u) = 40.8 \pm 3.2$ [39]. This implies that we really must include electromagnetic effects in our analysis as well. The electromagnetically-induced phase shifts appear to be small [15], so that we merely

include the modifications to the amplitudes themselves [40]. Following Ref. [15], we have

$$\begin{aligned}
\delta A^{em}_{1/2} &= \sqrt{2} C_{em} C g_8 \left(\frac{2}{3} C_{+-} + \frac{1}{3} C_{00} \right) \\
\delta A^{em}_{3/2} &= \frac{2}{5} C_{em} C g_8 \left(\frac{2}{3} (C_{+-} - C_{00}) + C_{+0} \right) \\
\delta A^{em}_{5/2} &= \frac{2}{5} C_{em} C g_8 \Big(C_{+-} - C_{00} - C_{+0} \Big)
\end{aligned} \tag{41}$$

where $C_{em} = (f_\pi/f_K)(\alpha/4\pi)(1+2\hat{m}/(m_s-\hat{m}))$ and the "dispersive matching" approach of Ref. [15] yields $C_{+-} = 14.8 \pm 3.5$, $C_{00} = 1.8 \pm 2.1$, and $C_{+0} = -7.1 \pm 7.4$. In the numerical estimates we use $2\hat{m}/(m_s-\hat{m}) = (m^2_{\pi^0}+m^2_{\pi^+})/(m^2_{K^0}+m^2_{K^+}-(m^2_{\pi^0}+m^2_{\pi^+}))$. Only electromagnetic effects associated with $(8_L, 1_R)$ transitions have been considered, as the $|\Delta I| = 1/2$ rule suggests they ought dominate. Including electromagnetic effects thus yields

$$\begin{aligned}
x &= \frac{\sqrt{2} r^{(3/2)}}{1 + r^{(1/2)}} \left(1 - \frac{2}{3\sqrt{3}} \epsilon_8 \frac{(2 + 3(r^{(3/2)} - r^{(1/2)})}{1 + r^{(1/2)}} - \frac{C_{em}(2C_{+-} + C_{00})}{3(1 + r^{(1/2)})} \right) \\
&+ \frac{\epsilon_8}{15} \sqrt{\frac{2}{3}} \frac{(10 - 9r^{(3/2)} + 60 r^{(1/2)})}{1 + r^{(1/2)}} + \frac{\sqrt{2}}{5} \frac{C_{em}(2(C_{+-} - C_{00}) + 3C_{+0})}{3(1 + r^{(1/2)})}
\end{aligned} \tag{42}$$

and

$$y = \frac{\sqrt{2}}{5} \left(\frac{\sqrt{3} \epsilon_8 r^{(3/2)} + C_{em}(C_{+-} - C_{00} - C_{+0})}{1 + r^{(1/2)}} \right) . \tag{43}$$

Using Ref. [15] we have $C_{em}(C_{+-} - C_{00} - C_{+0}) = 0.0029 \pm 0.0019$, as $f_K/f_\pi = 1.23 \pm 0.02$ [41]. Consequently if $r^{(3/2)}$ were as small as the isospin-symmetric limit would imply, then y ought be given by $C_{em}(C_{+-} - C_{00} - C_{+0})$, yet they are of opposite sign. This implies that the error in the $\delta_0 - \delta_2$ phase shift is even larger, or that the errors in the calculations of the electromagnetic effects are underestimated. Nevertheless, as apparently y is negative and $C_{em}(C_{+-} - C_{00} - C_{+0})$ is positive, the discrepancy could be resolved by adjusting $r^{(1/2)}$ and $r^{(3/2)}$ to suit the empirically determined x and y. Let us examine this point explicitly. In Table 2 we show the values of $r^{(1/2)}$ and $r^{(3/2)}$ which emerge from fitting the values of x and y which result from the empirical branching ratios and various values of the $\delta_0 - \delta_2$ phase shift difference.

The salient points of our analysis can be summarized as follows.

- If the $SU(3)_f$ relation $r^{(1/2)} = r^{(3/2)}/5$ is imposed, then the value of ϵ_8 which emerges is $\mathcal{O}(20\%)$ and is thus untenably large.

- If the $SU(3)_f$ relation $r^{(1/2)} = r^{(3/2)}/5$ is no longer imposed, and ϵ_8 is fixed as per $\epsilon_8 = 0.0106 \pm 0.0008$ [39], then $r^{(1/2)}$ is *very* different from $r^{(3/2)}/5$ — the $SU(3)_f$ breaking effects

are extremely large. This result is driven by large difference between the empirical value of y and the computed electromagnetic contribution [15]. That is, if we were to drop terms of $\mathcal{O}(r^{(3/2)}\epsilon_8)$ all together, then, with $\delta_0 - \delta_2 = 45°$, Eq. (43) implies that $r^{(1/2)} = -1.440$ and Eq. (42) implies that $r^{(3/2)} = 0.0184$. The inclusion of $\mathcal{O}(r^{(3/2)}\epsilon_8)$ terms do not significantly reduce this difficulty. Such large $\mathrm{SU}(3)_f$ breaking effects are difficult to reconcile with chiral power counting and model estimates, which suggest such effects are no more than 30% [42].

- The value of A_2/A_0 is generally different from and rather more uncertain than that which emerges from Eq. (30) in the isospin-symmetric limit, namely $|A_2/A_0| \approx 0.045$ with $A_2/A_0 > 0$.

The breaking of $\mathrm{SU}(3)_f$ relation $r^{(1/2)} = r^{(3/2)}/5$ apes the inclusion of higher order effects in the weak chiral Lagrangian, and the large breaking effects seen suggest that including $\mathcal{O}(p^4)$ effects are very important. This has some precedent, as in the isospin-symmetric limit, Ref. [14] finds a 30% quenching of the $\mathcal{O}(p^2)$ g_8 result in $\mathcal{O}(p^4)$. The $\mathrm{SU}(3)_f$ breaking effects seen, however, are much too large [42] and prompt an investigation of the presence of higher-order effects in a more systematic fashion.

We wish to consider how $\mathcal{O}(p^4)$ effects impact the parametrization of Eq. (37). We enlarge our parametrization by considering how the terms of the $\mathcal{O}(p^4)$ weak, chiral Lagrangian of Ref. [43] may be reorganized into the form of Eq. (38). We distinguish the $\mathcal{O}(m_d - m_u)$ terms which arise from "kinematics," i.e., from factors of m_{K^0}, from π^0-η mixing, as well as from the counterterms of the $\mathcal{O}(p^4)$, weak, chiral Lagrangian. We find that the effects of the higher-order terms can be absorbed in this case into *effective* g_8, $g_{27}^{(1/2)}$, and $g_{27}^{(3/2)}$ constants, with one additional phenomenological amplitude $\delta\tilde{A}_{5/2}^{\mathrm{h.o.}}$, generated by $\mathcal{O}(p^4)$ contributions of $(27_L, 1_R)$ character times $B_0(m_d - m_u)$. Varying the possible inputs within the bounds suggested by dimensional analysis, we are unable to reduce the $\mathrm{SU}(3)_f$ breaking of the relation $r^{(1/2)} = r^{(3/2)}/5$ to the level needed if the additional phenomenological $\delta\tilde{A}_{5/2}^{\mathrm{h.o.}}$ is generated solely by $m_d \neq m_u$ effects. Thus we are unable to construct a suitable phenomenological description of the $K \to \pi\pi$ amplitudes with the $\delta_0 - \delta_2$ phase shift of Ref. [2] and with the computed electromagnetic effects of Ref. [15]. The size of $\delta\tilde{A}_{5/2}^{\mathrm{h.o.}}$ required to generate suitably small violations of $r^{(1/2)} = r^{(3/2)}/5$ suggests the presence of missing electromagnetic effects generated by $(8_L, 1_R)$ operators. The authors of Ref. [15] are in the process of estimating additional electromagnetic effects [44]. The details of our efforts are delineated in the Appendix. Note that issues of a similar ilk have been addressed in Ref. [45].

It is worth noting that the conundrum we have been unable to resolve is unlikely to be due to "new" physics in $K \to \pi\pi$ decays. The operator-product expansion for $s \to d\bar{q}q$ transitions starts in dimension six, so that at most three u, d quark fields are present, implying that the short-distance operators generate at most a $|\Delta I| = 3/2$ transition. In next-to-leading order, as many as five u, d quark fields are present, so that a short-distance $|\Delta I| = 5/2$ transition is possible. Thus to estimate the plausibility of physics beyond the Standard Model as a source of $|\Delta I| = 5/2$ effects, we

need only estimate the relative importance of dimension-nine to dimension-six operators. Each new dimension is suppressed by the scale Λ — in the Standard Model, $\Lambda \sim M_W$, otherwise $\Lambda > M_W$. For $K \to \pi\pi$ decays the relative importance of the dimension-nine operators is no larger than $(M_K/M_W)^3$. Clearly short-distance physics cannot generate an appreciable $|\Delta I| = 5/2$ amplitude, so that the presence of physics beyond the Standard Model cannot be invoked to reconcile our difficulty.

The presence of a $|\Delta I| = 5/2$ amplitude also impacts the theoretical value of ϵ'/ϵ , for standard practice employs a value of ω determined from the $K \to \pi\pi$ branching ratios under the assumption that isospin symmetry is perfect. We proceed to investigate how the presence of a $|\Delta I| = 5/2$ amplitude impacts the value of ϵ'/ϵ .

5 Isospin Violation in Re (ϵ'/ϵ)

We wish to examine how isospin-violating effects impact the theoretical value of $\mathrm{Re}\,(\epsilon'/\epsilon)$ and the extraction of the value of ω, namely the ratio $\mathrm{Re}\,A_2/\mathrm{Re}\,A_0$, where A_I denotes the amplitude for $K \to (\pi\pi)_I$ and $(\pi\pi)_I$ denotes a $\pi\pi$ final state of isospin I. The empirical value of $\mathrm{Re}\,(\epsilon'/\epsilon)$ is inferred from the following ratio of ratios [46, 47]:

$$\mathrm{Re}\,(\frac{\epsilon'}{\epsilon}) = \frac{1}{6}\left[\left|\frac{\eta_{+-}}{\eta_{00}}\right|^2 - 1\right] , \tag{44}$$

where

$$\eta_{+-} \equiv \frac{\langle\pi^+\pi^-|\mathcal{H}_W|K_L^0\rangle}{\langle\pi^+\pi^-|\mathcal{H}_W|K_S^0\rangle} \quad ; \quad \eta_{00} \equiv \frac{\langle\pi^0\pi^0|\mathcal{H}_W|K_L^0\rangle}{\langle\pi^0\pi^0|\mathcal{H}_W|K_S^0\rangle} \tag{45}$$

and $\mathcal{H}_W$ is the effective weak Hamiltonian for kaon decays. Writing K_S^0 and K_L^0 in terms of the CP eigenstates $|K_\pm^0\rangle$ yields $|K_{L,S}^0\rangle = (|K_\mp^0\rangle + \bar\varepsilon|K_\pm^0\rangle)/\sqrt{1+|\bar\varepsilon|^2}$, noting that $|K_\pm^0\rangle = (|K^0\rangle \mp |\overline{K}^0\rangle)/\sqrt{2}$. Using Eq. (3) and treating the weak phases as small, so that only leading-order terms in ξ_0, ξ_2, ξ_{00}, and ξ_{+-} are retained, we find

$$\eta_{+-} = \epsilon + i\frac{\frac{1}{\sqrt{2}}|\frac{A_2}{A_0}|(\xi_2-\xi_0)e^{i(\delta_2-\delta_0)} + |\frac{A_{\rm IB}^{+-}}{A_0}|(\xi_{+-}-\xi_0)e^{i(\delta_{+-}-\delta_0)}}{1+\frac{1}{\sqrt{2}}|\frac{A_2}{A_0}|e^{i(\delta_2-\delta_0)} + |\frac{A_{\rm IB}^{+-}}{A_0}|e^{i(\delta_{+-}-\delta_0)}} \tag{46}$$

and

$$\eta_{00} = \epsilon - i\frac{\sqrt{2}|\frac{A_2}{A_0}|(\xi_2-\xi_0)e^{i(\delta_2-\delta_0)} - |\frac{A_{\rm IB}^{00}}{A_0}|(\xi_{00}-\xi_0)e^{i(\delta_{00}-\delta_0)}}{1-\sqrt{2}|\frac{A_2}{A_0}|e^{i(\delta_2-\delta_0)} + |\frac{A_{\rm IB}^{00}}{A_0}|e^{i(\delta_{00}-\delta_0)}} , \tag{47}$$

where $\epsilon \equiv \bar{\varepsilon} + i\xi_0$. Defining

$$\frac{\eta_{+-}}{\eta_{00}} \equiv 1 + 3\frac{\epsilon'}{\epsilon} \tag{48}$$

and retaining the leading terms in $|A_{\rm IB}^{+-}/A_0|$, $|A_{\rm IB}^{00}/A_0|$, and weak phases, we have

$$\begin{aligned}\frac{\epsilon'}{\epsilon} &= \frac{ie^{i(\delta_2-\delta_0-\Phi_\epsilon)}}{\sqrt{2}|\epsilon|}\Bigg[\left|\frac{A_2}{A_0}\right|(\xi_2-\xi_0)\left[1+\frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\left|\frac{A_2}{A_0}\right| - \frac{1}{3}\left(e^{i(\delta_{+-}-\delta_0)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right| + 2e^{i(\delta_{00}-\delta_0)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\right)\right] \\ &\quad + \frac{\sqrt{2}}{3}\left[e^{i(\delta_{+-}-\delta_2)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right|(\xi_{+-}-\xi_0) - e^{i(\delta_{00}-\delta_2)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|(\xi_{00}-\xi_0)\right] \\ &\quad - \frac{1}{3}\frac{|A_2|}{|A_0|}\left[e^{i(\delta_{+-}-\delta_0)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right|(\xi_{+-}-\xi_0) + 2e^{i(\delta_{00}-\delta_0)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|(\xi_{00}-\xi_0)\right]\Bigg] ,\end{aligned} \tag{49}$$

where we have retained terms of $\mathcal{O}(|A_2/A_0|^2)$ as well, for consistency. Note that $\epsilon = |\epsilon|e^{i\Phi_\epsilon}$. Equation (48) is consistent with the empirical definition of Eq. (44) as corrections of $(\epsilon'/\epsilon)^2$ are trivial. Alternatively,

$$\begin{aligned}\frac{\epsilon'}{\epsilon} &= -\frac{i\xi_0\omega e^{i(\delta_2-\delta_0-\Phi_\epsilon)}}{\sqrt{2}|\epsilon|}\Bigg(1 - \frac{1}{\omega}\Bigg(\left|\frac{A_2}{A_0}\right|\frac{\xi_2}{\xi_0}\left[1+\frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\left|\frac{A_2}{A_0}\right| - \frac{1}{3}\left[e^{i(\delta_{+-}-\delta_0)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right| + 2e^{i(\delta_{00}-\delta_0)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\right]\right] \\ &\quad + \frac{\sqrt{2}}{3}\left[e^{i(\delta_{+-}-\delta_2)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right|\frac{\xi_{+-}}{\xi_0} - e^{i(\delta_{00}-\delta_2)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\frac{\xi_{00}}{\xi_0}\right] \\ &\quad - \frac{1}{3}\frac{|A_2|}{|A_0|}\left[e^{i(\delta_{+-}-\delta_0)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right|\frac{\xi_{+-}}{\xi_0} + 2e^{i(\delta_{00}-\delta_0)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\frac{\xi_{00}}{\xi_0}\right]\Bigg)\Bigg) ,\end{aligned} \tag{50}$$

where

$$\begin{aligned}\omega &= \left|\frac{A_2}{A_0}\right| + \frac{\sqrt{2}}{3}\left(e^{i(\delta_{+-}-\delta_2)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right| - e^{i(\delta_{00}-\delta_2)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\right) + \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\left|\frac{A_2}{A_0}\right|^2 \\ &- \frac{2}{3}\left|\frac{A_2}{A_0}\right|\left(e^{i(\delta_{+-}-\delta_0)}\left|\frac{A_{\rm IB}^{+-}}{A_0}\right| + 2e^{i(\delta_{00}-\delta_0)}\left|\frac{A_{\rm IB}^{00}}{A_0}\right|\right) .\end{aligned} \tag{51}$$

Thus, working to leading order in isospin violation and ignoring electromagnetic effects in the "strong" phases, specifically implying as per Eqs. (3) and (25) that

$$\begin{aligned}A_{\rm IB}^{+-}e^{i\delta_{+-}} &= \delta A_{1/2}e^{i\delta_0} + \frac{1}{\sqrt{2}}\left(\delta A_{3/2}+\delta A_{5/2}\right)e^{i\delta_2} \\ A_{\rm IB}^{00}e^{i\delta_{00}} &= \delta A_{1/2}e^{i\delta_0} - \sqrt{2}\left(\delta A_{3/2}+\delta A_{5/2}\right)e^{i\delta_2} ,\end{aligned} \tag{52}$$

Eqs. (50,52) become

$$
\begin{aligned}
\frac{\epsilon'}{\epsilon} = -\frac{i\xi_0\omega e^{i(\delta_2-\delta_0-\Phi_\epsilon)}}{\sqrt{2}|\epsilon|}\Bigg(1 \quad &- \quad \frac{1}{\omega}\Bigg(\left|\frac{A_2}{A_0}\right|\frac{\xi_2}{\xi_0} + \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\left|\frac{A_2}{A_0}\right|^2\frac{\xi_2}{\xi_0} + \frac{\mathrm{Im}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|\xi_0} \\
&- \quad \left|\frac{A_2}{A_0}\right|\frac{\xi_2}{\xi_0}\left[\frac{\mathrm{Re}\,\delta A_{1/2}}{|A_0|} - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\frac{\mathrm{Re}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|}\right] \\
&- \quad \left|\frac{A_2}{A_0}\right|\left[\frac{\mathrm{Im}\,\delta A_{1/2}}{|A_0|\xi_0} - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\frac{\mathrm{Im}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|\xi_0}\right]\Bigg)\Bigg)\,,
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
\omega \quad = \quad & \left|\frac{A_2}{A_0}\right| + \frac{\mathrm{Re}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|} + \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\left|\frac{A_2}{A_0}\right|^2 + \\
- \quad & 2\left|\frac{A_2}{A_0}\right|\left[\frac{\mathrm{Re}\,\delta A_{1/2}}{|A_0|} - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\frac{\mathrm{Re}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|}\right]\,.
\end{aligned}
\tag{54}
$$

We can recast these formulae into a more familiar form [7] by writing Eq. (53) as

$$
\frac{\epsilon'}{\epsilon} = -\frac{i\omega e^{i(\delta_2-\delta_0-\Phi_\epsilon)}}{\sqrt{2}|\epsilon|\mathrm{Re}\,A_0}\left\{\mathrm{Im}\,A_0(1-\Omega_{\mathrm{IB}}) - \frac{1}{\omega}\mathrm{Im}\,A_2 - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\mathrm{Im}\,A_2\right\}\,,
\tag{55}
$$

where ω is defined by Eq. (54) and

$$
\begin{aligned}
\Omega_{\mathrm{IB}} \quad = \quad & \frac{1}{\omega}\Bigg(\frac{\mathrm{Im}(\delta A_{3/2}+\delta A_{5/2})}{\mathrm{Im}A_0} - \frac{\mathrm{Im}A_2}{\mathrm{Im}A_0}\left[\frac{\mathrm{Re}\,\delta A_{1/2}}{|A_0|} - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\frac{\mathrm{Re}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|}\right] \\
- \quad & \left|\frac{A_2}{A_0}\right|\left[\frac{\mathrm{Im}\,\delta A_{1/2}}{\mathrm{Im}A_0} - \frac{1}{\sqrt{2}}e^{i(\delta_2-\delta_0)}\frac{\mathrm{Im}(\delta A_{3/2}+\delta A_{5/2})}{\mathrm{Im}A_0}\right]\Bigg)\,.
\end{aligned}
\tag{56}
$$

If we assume that the $|\Delta I| = 1/2$ enhancement observed in $\mathrm{Re}\,A_I$ is germane to $\mathrm{Im}\,A_I$ as well, so that both $\mathrm{Re}\,A_0 \gg \mathrm{Re}\,A_2$ and $\mathrm{Im}\,A_0 \gg \mathrm{Im}\,A_2$ are satisfied, then if we ignore terms of $\mathcal{O}((\mathrm{Re}\,A_2/\mathrm{Re}\,A_0)(\epsilon_8,\alpha))$ and of $\mathcal{O}((\mathrm{Im}\,A_2/\mathrm{Im}\,A_0)(\epsilon_8,\alpha))$, as well as of $\mathcal{O}((|A_2|/|A_0|)^2)$, we find that Eq. (55) can be written as [7]

$$
\frac{\epsilon'}{\epsilon} = -\frac{i\omega e^{i(\delta_2-\delta_0-\Phi_\epsilon)}}{\sqrt{2}|\epsilon|\mathrm{Re}\,A_0}\left\{\mathrm{Im}\,A_0(1-\Omega_{\mathrm{IB}}) - \frac{1}{\omega}\mathrm{Im}\,A_2\right\}\,,
\tag{57}
$$

with

$$
\Omega_{\mathrm{IB}} = \frac{1}{\omega}\left(\frac{\mathrm{Im}(\delta A_{3/2}+\delta A_{5/2})}{\mathrm{Im}A_0}\right)
\tag{58}
$$

and

$$
\omega = \left|\frac{A_2}{A_0}\right| + \frac{\mathrm{Re}(\delta A_{3/2}+\delta A_{5/2})}{|A_0|}\,.
\tag{59}
$$

Equation (58) is proportional to $\mathrm{Im}A_{K^0\to\pi^+\pi^-} - \mathrm{Im}A_{K^0\to\pi^0\pi^0}$ and is generated by $(8_L, 1_R)$ operators. It is equivalent to Eq.(4) in Ref. [11].

In standard practice, the value of ω is typically extracted from the analysis of $K \to \pi\pi$ branching ratios in the isospin-perfect limit; specifically, ω is set equal to the RHS of Eq. (30), yielding [27]

$$2\sqrt{\frac{R_2}{3}} \equiv \omega_{\rm exp} = 0.0449 \pm 0.0003 \; . \tag{60}$$

From Eqs. (30,31), we see that ω as defined by Eq. (54) is actually given by

$$\omega = \omega_{\rm exp} + \frac{5}{3}y + \frac{1}{\sqrt{2}}(x+y)^2 - 2\frac{A_2}{A_0}\frac{\mathrm{Re}\,\delta A_{1/2}}{A_0} \; , \tag{61}$$

where we ignore terms of non-leading order in isospin violation, as well as terms of $\mathcal{O}((|A_2|/|A_0|)^2, (\alpha, \epsilon_8))$. If $\delta_0 - \delta_2 = 45^\circ \pm 6^\circ$ [2, 3], then we find from Table 1 that the second term of Eq. (61) is ~ -0.0110, whereas the third term is ~ 0.0008. We estimate the last term of Eq. (61) to be $\sim \pm 2(0.045)(0.01) \sim \pm 0.0010$. Thus the last two terms are small relative to the error in y — dropping them all together, we find [48]

$$\omega = 0.0339 \pm 0.0056 \; . \tag{62}$$

The use of the value of ω given in Eq. (62) tends to decrease the SM prediction of ϵ'/ϵ , both by an overall factor of $\sim 25\%$, as well as by enhancing the cancellation of the $\mathrm{Im}\,A_0$ and $\mathrm{Im}\,A_2$ contributions of Eq. (57). Note that our explicit estimate of the additional terms included in Eq. (54) suggests that the formulae of Eqs. (57), (58), and (59) characterize the isospin-violating contributions in a sufficiently accurate manner. In order to assess the impact of our numerical estimate of Eq. (62), let us turn to the schematic formula [6]

$$\frac{\epsilon'}{\epsilon} = 13\,\mathrm{Im}\lambda_t \left[B_6^{(1/2)}(1 - \Omega_{\eta+\eta'}) - 0.4\, B_8^{(3/2)} \right] \; , \tag{63}$$

in which $B_6^{(1/2)} = 1.0$, $B_8^{(3/2)} = 0.8$, $\mathrm{Im}\lambda_t = 1.3 \cdot 10^{-4}$, $\Omega_{\eta+\eta'} = 0.25$, and $\omega = 0.045$ yields the "central" SM value of $\epsilon'/\epsilon \sim 7 \cdot 10^{-4}$ [6]. Using Eq. (62) yields $\epsilon'/\epsilon \sim 4 \cdot 10^{-4}$, a 40% decrease. It has been recently suggested that isospin-breaking effects in the hadronization of the gluonic penguin operator can generate isospin-breaking contributions to $\Omega_{\eta+\eta'}$ beyond π^0-η, η' mixing, hence $\Omega_{\eta+\eta'} \to \Omega_{\rm IB}$ [11]. Interestingly, the use of the correct value of ω, Eq. (62), partially offsets the large increase in ϵ'/ϵ found in Ref. [11]. Using the estimate $\Omega_{\rm IB} \to -0.05 \to -0.78$ [11, 49], based exclusively on strong-interaction isospin breaking, we find with Eqs. (62) and (63) that

$$\frac{\epsilon'}{\epsilon} \sim (8 - 17) \cdot 10^{-4} \tag{64}$$

rather than

$$\frac{\epsilon'}{\epsilon} \sim (12 - 25) \cdot 10^{-4} \tag{65}$$

with $\omega = 0.045$ and Eq. (63). We anticipate that electromagnetic effects also contribute to $\Omega_{\rm IB}$, so that our numerical estimates are certainly incomplete, though indicative of the irreducible uncertainties present.

It is useful to contrast the relations we have found for ϵ'/ϵ , ω, and $\Omega_{\rm IB}$ with those used previously. Earlier treatments of strong-interaction isospin violation [8, 9, 10] considered π^0-η, η' mixing exclusively, as this is the only manner in which relevant $m_u \neq m_d$ effects appear in the $\mathrm{O}(p^2, 1/N_c)$ weak chiral Lagrangian. The η' enters as an explicit degree of freedom in these treatments [9, 10]. The small value of $\omega_{\rm exp}$ suggests that $(8_L, 1_R)$ operators dominate the isospin-violating contributions as well, and isospin violation based on the $(27_L, 1_R)$ contributions is thus neglected entirely. Assuming $(8_L, 1_R)$ operators dominate the isospin-violating effects means implicitly that the terms of $\mathcal{O}((\mathrm{Re}\, A_2/\mathrm{Re}\, A_0)(\epsilon_8, \alpha))$ and of $\mathcal{O}((\mathrm{Im}\, A_2/\mathrm{Im}\, A_0)(\epsilon_8, \alpha))$, as well as of $\mathcal{O}((|A_2|/|A_0|)^2)$, are all neglected. In the notation of Eq. (3) $A_{\rm IB}^{+-} = 0$ and $A_{\rm IB}^{00} = 2(\varepsilon_\eta \langle \pi^0 \eta | \mathcal{H}_W^8 | K^0 \rangle + \varepsilon_{\eta'} \langle \pi^0 \eta' | \mathcal{H}_W^8 | K^0 \rangle)$, where $\varepsilon_\eta, \varepsilon_{\eta'} \propto (m_d - m_u)$ and $\mathcal{L}_W^8$ denotes the effective weak Lagrangian transforming as $(8_L, 1_R)$ under $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ symmetry — $\mathcal{L}_W^8$ contains exactly one term. In Refs. [9, 10], the π^0-η, η' mixing contribution is incorporated by defining new I=0 and I=2 amplitudes, such that the form of the isospin decomposition of Eq. (2) is retained. Introducing $\Delta A_{0,2} \equiv A_{0,2} - A_{0,2}^{(0)}$ to describe the change in the $I = 0$ and $I = 2$ amplitudes under this procedure we find

$$\Delta A_2 = -\frac{\sqrt{2}}{3} A_{\rm IB}^{00} \quad ; \quad \Delta A_0 = \frac{1}{3} A_{\rm IB}^{00} \, . \tag{66}$$

Thus one recovers the *form* of Eq. (57) with $\delta A_{3/2} = \delta A_{5/2} = 0$. Rewriting the imaginary parts in terms of the isospin-perfect pieces $\mathrm{Im}\, A_I^{(0)}$, i.e., in the absence of π^0-η, η' mixing, yields [7]

$$\frac{\epsilon'}{\epsilon} = -\frac{i e^{i(\delta_2 - \delta_0 - \Phi_\epsilon)}}{\sqrt{2} |\epsilon| \mathrm{Re}\, A_0} \left\{ \omega\, \mathrm{Im}\, A_0^{(0)} (1 - \Omega_{\eta + \eta'}) - \mathrm{Im}\, A_2^{(0)} \right\} \tag{67}$$

with

$$\begin{aligned} \Omega_{\eta+\eta'} \quad &= \frac{1}{\mathrm{Im}\, A_0^{(0)}} \left(\frac{\mathrm{Im}\, \Delta A_2}{\omega} - \mathrm{Im}\, \Delta A_0 \right) \\ &\simeq \frac{1}{\omega} \frac{\mathrm{Im}\, \Delta A_2}{\mathrm{Im}\, A_0^{(0)}} \, , \end{aligned} \tag{68}$$

noting that only the ΔA_2 term is retained for phenomenological purposes [7]. Equation (67) results from absorbing the isospin-violating contributions into two amplitudes, “A_0” and “A_2”. A third amplitude is permitted in the presence of isospin violation. However, if we neglect electromagnetic

effects and consider isospin violation based on $(8_L, 1_R)$ operators only, then only two amplitudes are present, and the above procedure is appropriate. Equation (53) requires no such assumptions and thus is more general than the expression in Eq. (67). Let us now consider Eq. (57) in the event π^0-η, η' mixing were the only source of isospin-violation present — we will continue to assume that $(8_L, 1_R)$ transitions generate the only numerically important isospin-violating effects. Note that the "kinematic" $m_d \neq m_u$ effect from $m_{K^0}^2$ does not contribute to $\delta A_{3/2} + \delta A_{5/2}$ in this case. The mixing parameters ϵ_η and $\epsilon_{\eta'}$ are real [22], so that $\mathrm{Im}(\delta A_{3/2} + \delta A_{5/2})$ is determined by $\langle \eta^{(\prime)} | \mathcal{L}_W^8 | K^0 \rangle$. The Lagrangian $\mathcal{L}_W^8$ contains exactly one term, so that the matrix elements are proportional to $A_0^{(0)}$, and the proportionality constant is real. Thus $\mathrm{Im}(\delta A_{3/2} + \delta A_{5/2})/\mathrm{Im}\, A_0 = \mathrm{Re}(\delta A_{3/2} + \delta A_{5/2})/\mathrm{Re}\, A_0$ as π^0-η, η' mixing is real [22], so that we have

$$\frac{\epsilon'}{\epsilon} = -\frac{i\xi_0 \omega e^{i(\delta_2 - \delta_0 - \Phi_\epsilon)}}{\sqrt{2}|\epsilon|} \left(1 - \frac{1}{\omega} \left(|\frac{A_2}{A_0}| \frac{\xi_2}{\xi_0} - \frac{\sqrt{2}}{3} |\frac{A_{\mathrm{IB}}^{00}}{A_0}| \right) \right) \; . \tag{69}$$

Using Eq. (59) we find

$$\begin{aligned} \frac{\epsilon'}{\epsilon} &= -\frac{i\xi_0 e^{i(\delta_2 - \delta_0 - \Phi_\epsilon)}}{\sqrt{2}|\epsilon|} \left(|\frac{A_2}{A_0}| - \frac{\sqrt{2}}{3} |\frac{A_{\mathrm{IB}}^{00}}{A_0}| - \left(|\frac{A_2}{A_0}| \frac{\xi_2}{\xi_0} - \frac{\sqrt{2}}{3} |\frac{A_{\mathrm{IB}}^{00}}{A_0}| \right) \right) & (70) \\ &= -\frac{i\xi_0 e^{i(\delta_2 - \delta_0 - \Phi_\epsilon)}}{\sqrt{2}|\epsilon|} |\frac{A_2}{A_0}| \left(1 - \frac{\xi_2}{\xi_0} \right) & (71) \end{aligned}$$

and thus the inclusion of isospin-violating effects in $\mathcal{O}(p^2)$ acts to correct for isospin violation in the extraction of ω from $K \to \pi\pi$ branching ratios, to recover the "true" $|A_2|/|A_0|$. Equation (69) can be rewritten

$$\frac{\epsilon'}{\epsilon} = -\frac{i\omega e^{i(\delta_2 - \delta_0 - \Phi_\epsilon)}}{\sqrt{2}|\epsilon| \mathrm{Re}\, A_0} \left\{ \mathrm{Im}\, A_0^{(0)} (1 - \tilde{\Omega}_{\eta+\eta'}) - \frac{1}{\omega} \mathrm{Im}\, A_2^{(0)} \right\} \tag{72}$$

where

$$\tilde{\Omega}_{\eta+\eta'} = -\frac{\sqrt{2}}{3\omega} \frac{|A_{\mathrm{IB}}^{00}|}{|A_0|} \; . \tag{73}$$

This is identical to Eq. (67) as $\tilde{\Omega}_{\eta+\eta'} = \Omega_{\eta+\eta'}$. In $\mathcal{O}(p^4)$ this simple interpretation of isospin-violating contributions in Ω_{IB} as modifications of ω does not carry as $\xi_{+-} \neq \xi_{00} \neq \xi_0$ in general. The interpretation also fails if $(27_L, 1_R)$ operators are included in the description of isospin-violating effects.

6 Conclusions

We have established a framework for the analysis of $K \to \pi\pi$ decays in the presence of strong-interaction isospin violation, so that the "true" $|\Delta I| = 1/2$ and $|\Delta I| = 3/2$ amplitudes can be

assessed. In particular, using unitarity arguments, we have shown that Watson's theorem, namely, the parametrization of Eq. (25), is appropriate to $\mathcal{O}((m_d - m_u)^2)$ to all orders of chiral perturbation theory. If we accept, as per Ref. [15], that electromagnetic effects do not alter the structure of Eq. (25), we can enlarge our analysis of $K \to \pi\pi$ decays in $\mathcal{O}(m_d - m_u)$ to include electromagnetic effects as well. Incorporating the electromagnetic corrections of Ref. [15] and the $\delta_0 - \delta_2$ phase shift of Ref. [2], we are unable to fit the $K \to \pi\pi$ branching ratio data with effective $(8_L, 1_R)$ and $(27_L, 1_R)$ low-energy constants in the framework of chiral perturbation theory, as our fits require the existence of intolerably large, higher-order corrections. Our failure, in retrospect, is predicated by the observation that the empirical value of the $|\Delta I| = 5/2$ amplitude, determined by the value of the $\delta_0 - \delta_2$ phase shift, is much larger and of opposite sign to the electromagnetically generated $|\Delta I| = 5/2$ amplitude computed by Ref. [15] in either chiral perturbation theory or in their dispersive matching approach. Although our results suggest that our phenomenological analysis is incomplete, that is, that missing electromagnetic effects likely exist, it is clear that the value of A_2/A_0 — the "true" ratio of the $|\Delta I| = 3/2$ to $|\Delta I| = 1/2$ amplitudes — is quite uncertain, as it is sensitive to the inclusion of isospin-violating effects.

Turning to an analysis of ϵ'/ϵ in the presence of isospin violation, and applying the parametrization of Eq. (25), we find that an empirical $|\Delta I| = 5/2$ amplitude of the magnitude we have found generates a significant decrease in the Standard Model prediction of ϵ'/ϵ — although this decrease has a considerable uncertainty, quantified through the errors in the $K \to \pi\pi$ branching ratios and the $\delta_0 - \delta_2$ phase shift.

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7 Appendix

We wish to consider how $\mathcal{O}(p^4)$ effects impact the parametrization of Eq. (37). We find by explicit calculation that the $\mathcal{O}(p^4)$ contributions of the weak, chiral Lagrangian of Ref. [43] can be

reorganized into

$$\begin{aligned}
A_{K^0\to\pi^+\pi^-} &= \sqrt{2}Ci(1+\frac{2}{\sqrt{3}}\epsilon_1)\left(\tilde{g}_8+\tilde{g}_{27}^{(1/2)}+\tilde{g}_{27}^{(3/2)}+\frac{1}{2}\delta\tilde{A}_{5/2}^{\rm h.o.}\right)\\
A_{K^0\to\pi^0\pi^0} &= \sqrt{2}Ci(1+\frac{2}{\sqrt{3}}\epsilon_1)\left(\tilde{g}_8+\tilde{g}_{27}^{(1/2)}-2\tilde{g}_{27}^{(3/2)}-\frac{2\epsilon_2}{\sqrt{3}}(\tilde{g}_8+6\tilde{g}_{27}^{(1/2)}-3\tilde{g}_{27}^{(3/2)})-\delta\tilde{A}_{5/2}^{\rm h.o.}\right) \quad (74)\\
A_{K^+\to\pi^+\pi^0} &= Ci(1-\frac{2}{\sqrt{3}}\epsilon_1)\left(3\tilde{g}_{27}^{(3/2)}+\frac{\epsilon_2}{\sqrt{3}}(2\tilde{g}_8+12\tilde{g}_{27}^{(1/2)}+3\tilde{g}_{27}^{(3/2)})-\delta\tilde{A}_{5/2}^{\rm h.o.}\right)\ ,
\end{aligned}$$

where the effects of the higher-order weak counterterms are lumped into the effective constants $\tilde{g}_8$, $\tilde{g}_{27}^{(1/2)}$, $\tilde{g}_{27}^{(3/2)}$, and a new $|\Delta I|=5/2$ contribution $\delta\tilde{A}_{5/2}^{\rm h.o.}$, which is of order $D_iB_0(m_d-m_u)$, where D_i is a $\mathcal{O}(p^4)$ counterterm of $(27_L,1_R)$ character. Were $\delta\tilde{A}_{5/2}^{\rm h.o.}=0$ and $\epsilon_1=\epsilon_2=\epsilon_8$, we would recover the parametrization of Eq. (37). In Eq. (74), we have explicitly separated the strong-interaction isospin violation which emerges from meson mass differences, namely $m^2_{K^0,K^+}$, from that generated by $\pi^0-\eta$ mixing. The parameters ϵ_1 and ϵ_2 denote these two respective sources of isospin violation. Note that isospin-violating effects beyond π^0-η mixing, as discussed in Ref. [11], are embedded in $\tilde{g}_{27}^{(3/2)}$ and $\tilde{g}_{27}^{(1/2)}$. In $\mathcal{O}(p^2)$, ϵ_1 and ϵ_2 are given by $\sqrt{3}(m_d-m_u)/(4(m_s-\hat{m}))$. In $\mathcal{O}(p^4)$, ϵ_2 is modified by $\pi^0-\eta'$ mixing, as realized by the coefficients of the $\mathcal{O}(p^4)$ strong chiral Lagrangian [12]. Note that the cancellation of the $\epsilon_8 g_8$ contribution to the $K\to\pi^0\pi^0$ amplitude found in $\mathcal{O}(p^2)$ no longer occurs if $\epsilon_1\neq\epsilon_2$. Working consistently to $\mathcal{O}(m_d-m_u)$, and including electromagnetic effects, we find that Eq. (74) implies

$$\begin{aligned}
x = \frac{\sqrt{2}r^{(3/2)}}{1+r^{(1/2)}}&\left(1-\frac{2}{3\sqrt{3}}\frac{(3\epsilon_1-\epsilon_2+3r^{(3/2)}\epsilon_2+3r^{(1/2)}(\epsilon_1-2\epsilon_2))}{1+r^{(1/2)}}-\frac{h_1C_{em}(2C_{+-}+C_{00})}{3(1+r^{(1/2)})}\right) \quad (75)\\
&+\frac{1}{15}\sqrt{\frac{2}{3}}\frac{(10\epsilon_2-r^{(3/2)}(6\epsilon_1+3\epsilon_2)+60r^{(1/2)}\epsilon_2)}{1+r^{(1/2)}}+\frac{\sqrt{2}}{5}\frac{h_1C_{em}(2(C_{+-}-C_{00})+3C_{+0})}{3(1+r^{(1/2)})}
\end{aligned}$$

and

$$y=\frac{\sqrt{2}}{5}\left(\frac{\sqrt{3}r^{(3/2)}(4\epsilon_1-3\epsilon_2)+h_1C_{em}(C_{+-}-C_{00}-C_{+0})}{1+r^{(1/2)}}\right)+\frac{1}{\sqrt{2}}\frac{(\delta\tilde{A}_{5/2}^{\rm h.o.}/\tilde{g}_8)}{1+r^{(1/2)}}\ . \qquad (76)$$

We have defined $r^{(3/2)}\equiv\tilde{g}_{27}^{(3/2)}/\tilde{g}_8$ and $r^{(1/2)}\equiv\tilde{g}_{27}^{(1/2)}/\tilde{g}_8$, and the parameter $h_1\equiv g_8/\tilde{g}_8$. We estimate

$$\begin{aligned}
\frac{\delta\tilde{A}_{5/2}^{\rm h.o.}}{\tilde{g}_8} &\sim \left(\frac{\tilde{g}_{27}^{(3/2)}}{\tilde{g}_8}\right)\left(\frac{g_{27}^{(3/2)}}{\tilde{g}_{27}^{(3/2)}}\right)\left(\frac{4\epsilon_2}{\sqrt{3}}\right)\left(\frac{B_0(m_s-\hat{m})}{\Lambda^2_{\chi SB}}\right)\\
&\equiv (0.52)h_2r^{(3/2)}\epsilon_2\ ,
\end{aligned} \qquad (77)$$

where $B_0(m_s-\hat{m})/\Lambda^2_{\chi SB}\sim 0.23$. We expect the parameters h_1 and h_2 to be of order unity. Higher-order effects in the weak chiral Lagrangian serve to make $\tilde{g}_{27}^{(1/2)}\neq\tilde{g}_{27}^{(3/2)}/5$ — the term D_6, e.g.,

in the $\mathcal{O}(p^4)$ weak, chiral Lagrangian of Ref. [43] generates such an inequality. Consequently, we expect from dimensional analysis

$$\frac{\delta\tilde{g}_{27}^{1/2}}{\tilde{g}_{27}^{3/2}} \equiv \frac{\tilde{g}_{27}^{(1/2)} - \tilde{g}_{27}^{(3/2)}/5}{\tilde{g}_{27}^{(3/2)}} \sim \left(\frac{g_{27}^{(3/2)}}{\tilde{g}_{27}^{(3/2)}}\right)\left(\frac{B_0(m_s - \hat{m})}{\Lambda_{\chi SB}^2}\right) \equiv 0.23 h_3 \;, \tag{78}$$

where the parameter h_3 ought be of order unity. A model estimate of $\delta\tilde{g}_{27}^{1/2}/\tilde{g}_{27}^{3/2}$ suggests that it is less than 30% [42]. Isospin-violating contributions, ignored in Eq. (78), also contribute to $\delta\tilde{g}_{27}^{1/2}/\tilde{g}_{27}^{3/2}$; the largest terms are typified by $B_0(m_d - m_u)E_i$, where E_i is an $\mathcal{O}(p^4)$ counterterm of $(8_L, 1_R)$ in character, and thus generate, crudely, an additional $\sim$ 10% effect. The value of $(r^{(1/2)} - r^{(3/2)}/5)/r^{(3/2)}$ found in Table 2 far exceeds the estimate of Eq. (78). We thus wish to see whether plausible choices of h_1, h_2, and ϵ_2 can serve to reduce the $\mathrm{SU}(3)_f$ breaking of the relation $r^{(3/2)} = r^{(1/2)}/5$ found in Table 2 to a plausible level.

We explore how the values of $r^{(3/2)}$ and $r^{(1/2)}$ vary as a function of ϵ_2, h_1, and h_2 in Table 3. We fix $\epsilon_1 = 0.0106 \pm 0.0008$ [39] and choose $\delta_0 - \delta_2 = 51°$. The latter is determined by the central value of 45° given in Ref. [2] plus 6°, the $+1\sigma$ excursion permitted. We estimate that h_1 could be as small as 0.5, and we choose two different values for ϵ_2: we use the result determined from the $\mathcal{O}(p^4)$ strong chiral Lagrangian of Ref. [12] as well as the estimate $\epsilon_2 = 2\epsilon_1 \pm \epsilon_1$. The central value and its error assigned to ϵ_2 in this latter estimate is rather generous; we observe that electromagnetic effects, not included in Ref. [15], can enhance the π^0-η, η' mixing angle slightly [9]. Despite our efforts, a value of $h_2 \sim -25$ or larger is required to make the $\mathrm{SU}(3)_f$ breaking of $(r^{(1/2)} - r^{(3/2)}/5)/r^{(3/2)}$ no more than 100%. Interestingly, replacing the estimates of the electromagnetic corrections in the dispersive matching approach with those determined in chiral perturbative theory does increase the errors in the determined values of $r^{(3/2)}$ and $r^{(1/2)}$, but not sufficiently to reduce the value of h_2 substantially. It seems unlikely that strong-interaction isospin-violating effects can resolve the difference between the empirical value of y predicated by a phase shift $\delta_0 - \delta_2 \sim 45°$ and the electromagnetic effects computed in Ref. [15].

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[17] Note that the properly symmetrized state $|\pi^-\pi^+\rangle_{\rm sym} \equiv (|\pi_1^+\pi_2^-\rangle + |\pi_1^-\pi_2^+\rangle)/\sqrt{2} = \sqrt{2}|\pi^+\pi^-\rangle$ ought appear throughout. The extra factor of $\sqrt{2}$ in the definition of $|\pi^-\pi^+\rangle_{\rm sym}$ implies that the branching ratio into $K \to \pi^+\pi^-$ is a factor of 2 larger than suggested by Eq. (2).

[18] The parity of a three-π state, with coordinates $\vec{r}_1, \vec{r}_2$, and $\vec{r}_3$, is determined by the function $-(\vec{r}_2 - \vec{r}_1)^{l_1}(\vec{r}_3 - (\vec{r}_1 + \vec{r}_2)/2)^{l_2}$. For the $J = 0$ state, $l_1 = l_2 = l$, so that the $J = 0$ state is of odd parity.

[19] The only inelastic channel in the vicinity of $\sqrt{s} \sim m_K$ is the transition of $2\pi \to 4\pi$ with a threshold of $\sqrt{s} \simeq 0.56$ GeV; this reaction is also strongly phase-space suppressed near threshold, so that no significant inelasticity exists for $\sqrt{s} < 0.8$ GeV [3].

[20] In writing Eq. (8), we follow the form of the coupled-channel S-matrix for $J = 1$, $S = 1$ nucleon-nucleon scattering in the presence of a tensor force, note M. A. Preston and R. K. Bhaduri, *Structure of the Nucleus*, Addison-Wesley, Reading, MA, 1975, p. 141ff.

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[24] Equivalently, one can show that the Clebsch-Gordon coefficient for $(\pi\pi)_{I,I_3} \otimes (m_d - m_u)_{I=1,I_3=0} \Rightarrow (\pi\pi)_{I,I_3}$, where $I = 0, 2$, is identically zero.

[25] Neglecting the presence of electromagnetically generated terms in F in Eq. (5) predicates the structure of Eq. (15) on which our estimate is based. Thus our numerical estimate of Δ_2 assumes that these neglected terms are negligible relative to the electromagnetic effects we consider.

[26] In $B \to \pi\pi$ decays, the analogous inequality disrupts the isospin analysis proposed to obviate penguin "pollution" in the extraction of $\sin 2\alpha$ from the mixing-induced asymmetry in $B^0 \to \pi^+\pi^-$, see S. Gardner, *Phys. Rev.* **D59**, 077502 (1999); hep-ph/9906269. Note that in the "physical basis" defined in Eq. (9), the "η" component in the $\pi\pi$ final state gives rise to the $\delta A_{5/2}$ contributions – so that, in this basis, the extra amplitude is labelled "$I = 1$".

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Table 2: The values of $r^{(1/2)}$ and $r^{(3/2)}$ determined by fitting Eqs.(42) and (43) to the empirically determined x and y, resulting from the phase shift differences, $\delta_0-\delta_2$, compiled from various sources. We also show the values of $r^{(1/2)}$ and $r^{(3/2)}$ which result were the central value of $\delta_0 - \delta_2$ 1σ or 2σ larger than that reported by Ref. [2]. Electromagnetic effects are included as per Ref. [15]. Note that (C) and (D) denote the results as computed in chiral perturbation theory (C) and in the "dispersive matching" (D) approach, respectively. We also show the values of $r^{(1/2)}$ and $r^{(3/2)}$ which result if the electromagnetically-generated phase shift, $\gamma_2 \simeq 4.5^\circ$ [15], is included — using Ref. [2] this effectively implies $\delta_0 - \delta_2 = 40.5^\circ$. The parameter ϵ_8 is taken to be $\epsilon_8 = 0.0106 \pm 0.0008$ [39]. Solutions yielding $A_2/A_0 \gtrsim 1$ have been omitted. For comparison, note that the analysis of R_2 in the isospin perfect limit yields $|x| = |A_2/A_0| = 0.0449 \pm 0.0003$ [13].

$\delta_0 - \delta_2 = 40.5^\circ \pm 6^\circ$	$x = 0.0394 \pm 0.0018$	$y = -0.0082 \pm 0.0027$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0173 ± 0.0075	-1.32 ± 0.28	-0.078 ± 0.078	C
0.0176 ± 0.0079	-1.36 ± 0.19	-0.069 ± 0.046	D
$\delta_0 - \delta_2 = 42^\circ \pm 4^\circ$ [34]	$x = 0.0398 \pm 0.0016$	$y = -0.0077 \pm 0.0024$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0173 ± 0.0077	-1.34 ± 0.30	-0.073 ± 0.074	C
0.0175 ± 0.0081	-1.39 ± 0.20	-0.064 ± 0.043	D
$\delta_0 - \delta_2 = 45^\circ \pm 6^\circ$ [2]	$x = 0.0405 \pm 0.0021$	$y = -0.0066 \pm 0.0032$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0171 ± 0.0083	-1.39 ± 0.37	-0.061 ± 0.069	C
0.0172 ± 0.0087	-1.45 ± 0.28	-0.054 ± 0.044	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$x = 0.0424 \pm 0.0027$	$y = -0.0038 \pm 0.0041$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.015 ± 0.012	-1.69 ± 0.93	-0.032 ± 0.058	C
0.015 ± 0.013	-1.79 ± 0.91	-0.027 ± 0.045	D
$\delta_0 - \delta_2 = 57^\circ \pm 6^\circ$	$x = 0.0451 \pm 0.0037$	$y = 0.00027 \pm 0.0055$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.1 ± 1.8	10 ± 250	0.013 ± 0.064	C
0.1 ± 2.1	12 ± 290	0.013 ± 0.057	D

Table 3: The values of $r^{(1/2)}$ and $r^{(3/2)}$ determined by fitting Eqs.(75) and (76) to the empirically determined x and y, resulting from the phase shift difference, $\delta_0 - \delta_2$. Solutions yielding $A_2/A_0 \gtrsim 1$ have been omitted. The parameter $\epsilon_1 = 0.0106 \pm 0.0008$ [39] throughout. No errors are assigned to the h_1 and h_2 parameters. Note that (C) and (D) denote the electromagnetic corrections of Ref. [15] as computed in chiral perturbation theory (C) and in the "dispersive matching" (D) approach. The ratio $A_2/A_0 \equiv \sqrt{2}r^{(3/2)}/(1 + r^{(1/2)})$ does include $m_d \neq m_u$ effects through the absorbed $\mathcal{O}(p^4)$ counterterms.

$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 2\epsilon_1 \pm \epsilon_1$	$h_1 = 1 \quad h_2 = -1$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.044 ± 0.026	-1.49 ± 0.77	-0.13 ± 0.19	C
0.047 ± 0.027	-1.59 ± 0.72	-0.11 ± 0.14	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 2\epsilon_1 \pm \epsilon_1$	$h_1 = 0.5 \quad h_2 = -1$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.035 ± 0.016	-1.19 ± 0.35	-0.26 ± 0.44	C
0.037 ± 0.015	-1.24 ± 0.33	-0.22 ± 0.33	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 2\epsilon_1 \pm \epsilon_1$	$h_1 = 0.5 \quad h_2 = -25$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0296 ± 0.0092	-0.36 ± 0.48	0.065 ± 0.069	C
0.0303 ± 0.0086	-0.39 ± 0.44	0.070 ± 0.070	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 0.0133 \pm 0.0025$ [12]	$h_1 = 0.5 \quad h_2 = -25$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0289 ± 0.0064	-0.80 ± 0.27	0.20 ± 0.30	C
0.016 ± 0.011	-1.046 ± 0.090	-0.48 ± 0.62	C
0.0300 ± 0.0059	-0.83 ± 0.17	0.25 ± 0.28	D
0.0174 ± 0.0059	-1.063 ± 0.072	-0.39 ± 0.33	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 2\epsilon_1 \pm \epsilon_1$	$h_1 = 0.5 \quad h_2 = -50$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.0215 ± 0.0097	0.02 ± 0.57	0.030 ± 0.029	C
0.0220 ± 0.0093	0.00 ± 0.54	0.031 ± 0.029	D
$\delta_0 - \delta_2 = 51^\circ \pm 6^\circ$	$\epsilon_2 = 0.0133 \pm 0.0025$ [12]	$h_1 = 0.5 \quad h_2 = -50$	
$r^{(3/2)}$	$r^{(1/2)}$	A_2/A_0	em
0.026 ± 0.0013	-0.37 ± 0.72	0.058 ± 0.067	C
0.0260 ± 0.0016	-0.41 ± 0.63	0.063 ± 0.069	D